# A Linear Algorithm for Finding Total Colorings of Partial $k$-Trees (Algorithm Engineering as a New Paradigm)

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# 部分 $k$ 木を全彩色する線形時間アルゴリズム

## A Linear Algorithm for Finding Total Colorings of Partial $k$-Trees

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## Abstract

A total coloring of a graph $G$ is a coloring of all elements of $G$, i.e. vertices and edges, in such a way that no two adjacent or incident elements receive the same color. The total coloring problem is to find a total coloring of a given graph with the minimum number of colors. Many combinatorial problems can be efficiently solved for partial $k$-trees, i.e., graphs with bounded tree-width. However, no efficient algorithm has been known for the total coloring problem on partial $k$-trees although a polynomial-time algorithm of very high order has been known. In this paper, we give a linear-time algorithm for the total coloring problem on partial $k$-trees with bounded $k$.

**Keywords:** vertex-coloring, edge-coloring, total coloring, generalized coloring, partial $k$-tree

## 1 Introduction

A total coloring is a mixture of ordinary vertex-coloring and edge-coloring. That is, a *total coloring* of a graph $G$ is an assignment of colors to its vertices and edges so that no two adjacent vertices have the same color, no two adjacent edges have the same color, and no edge has the same color as one of its ends. The minimum number of colors required for a total coloring of a graph $G$ is called the *total chromatic number* of $G$, and denoted by $\chi_t(G)$. Figure 1 illustrates a total coloring of a graph $G$ using $\chi_t(G) = 4$ colors.

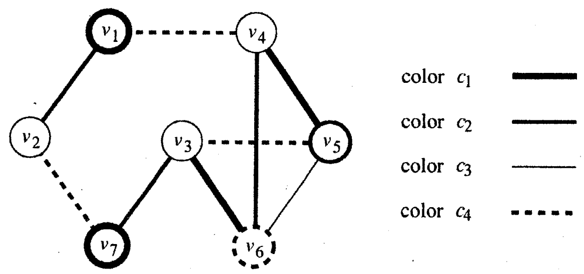


Figure 1: A total coloring of a graph with four colors.

This paper deals with the *total coloring problem* which asks to find a total coloring of a given graph $G$ using the minimum number $\chi_t(G)$ of colors. Since the problem is NP-complete for general graphs [Sán89], it is very unlikely that there exists an efficient algorithm for solving the problem for general graphs. On the other hand, many combinatorial problems including the vertex-coloring problem and the edge-coloring problem can be solved for partial $k$-trees with bounded $k$ very efficiently, mostly in linear time [ACPS93, AL91, BPT92, Cou90, CM93, ZSN96]. However, no efficient algorithm has been known for the total coloring problem on partial $k$-trees. Although the total coloring problem can be solved in polynomial time for partial $k$-trees by a dynamic programming algorithm, the time complexity $O(n^{2^{4(k+1)}+1})$ is very high [IZN99].

In this paper, we give a linear-time algorithm to solve the total coloring problem for partial $k$-trees with bounded $k$. The outline of the algorithm is as follows. For a given partial $k$-tree $G = (V, E)$, we first find an appropriate subset $F \subseteq E$ inducing a forest of $G$, then find a "generalized coloring" of $G$ for $F$ and an ordinary edge-coloring of the subgraph $H = G[\bar{F}]$ of $G$ induced by $\bar{F} = E - F$, and finally superimpose the edge-coloring on the generalized coloring to obtain a total coloring of $G$. The generalized coloring is an extended version of a total coloring and an ordinary vertex-coloring, and is newly introduced in this paper. Since $F$ induces a forest of $G$, a generalized coloring of $G$ for $F$ can be found in linear time. Since $H$ is a partial $k$-tree, an edge-coloring of $H$ can be found in linear time. Hence the total running time of our algorithm is linear. Thus our algorithm is completely different from an ordinary dynamic programming approach.

The paper is organized as follows. In Section 2, we give some basic definitions. In Section 3, we give a linear algorithm for finding a total coloring of a partial $k$-tree, and verify the correctness of the algorithm. Finally we conclude in Section 4 with a discussion of the results and some related works.

## 2 Terminologies and Definitions

In this section we give some basic terminologies and definitions.

For two sets $A$ and $B$, we denote by $A - B$ the set of elements $a$ such that $a \in A$ and $a \notin B$.

We denote by $G = (V, E)$ a simple undirected graph with a vertex set $V$ and an edge set $E$. For a graph $G = (V, E)$ we often write $V = V(G)$ and $E = E(G)$. We denote by $n$ the cardinality of $V(G)$. We denote by $\chi'(G)$ the minimum number of colors required for an ordinary edge-coloring of $G$, and call $\chi'(G)$ the *chromatic index of* $G$.

For a set $F \subseteq E$ and a vertex $v \in V$, we write $d_F(v, G) = |\{(v, w) \in F : w \in V\}|$ and $\Delta_F(G) = \max\{d_F(v, G) : v \in V\}$. In particular, we call $d(v, G) = d_E(v, G)$ the *degree* of $v$, and $\Delta(G) = \Delta_E(G)$ the *maximum degree* of $G$.

Let $F$ be a subset of $E$, called a *colored edge set*, and let $C$ be a set of colors. A *generalized coloring of a graph $G$ for $F$* is a mapping $f : V \cup F \to C$ satisfying the following three conditions:

(1) the restriction of the mapping $f$ to $V$ is a vertex-coloring of $G$, that is, $f(v) \neq f(w)$ for any pair of adjacent vertices $v$ and $w$ in $G$;

(2) the restriction of the mapping $f$ to $F$ is an edge-coloring of the subgraph $G[F]$ of $G$ induced by $F$, that is, $f(e) \neq f(e')$ for any pair of edges $e, e' \in F$ sharing a common end; and

(3) $f(v) \neq f(e)$ for any pair of a vertex $v \in V$ and an edge $e \in F$ incident to $v$.

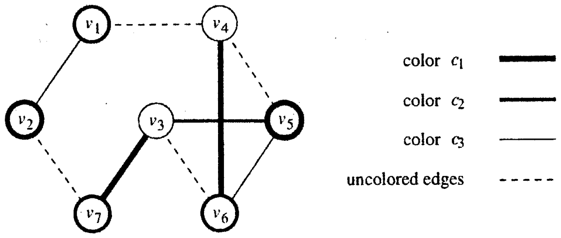


Figure 2: A generalized coloring of a graph with three colors.

Note that the edges in $\bar{F} = E - F$ are not colored by the generalized coloring $f$. We call the edges in $F$ *colored edges* and the edges in $\bar{F}$ *uncolored edges*. A *total coloring of* $G$ is a generalized coloring for a colored edge set $F = E$, while a vertex-coloring is a generalized coloring for a colored edge set $F = \emptyset$. Thus a generalized coloring is an extension of a total coloring and a vertex-coloring. It should be noted that a generalized coloring of $G$ for $F$ is a total coloring of $G[F]$ but a total coloring of $G[F]$ is not always a generalized coloring of $G$ for $F$. The minimum number of colors required for a generalized coloring of $G$ for $F$ is called the *generalized chromatic number of* $G$, and is denoted by $\chi_t(G, F)$. In particular, we denote $\chi_t(G, E)$ by $\chi_t(G)$, and call $\chi_t(G)$ the *total chromatic number of a graph* $G$. Clearly $\chi_t(G, F) \geq \Delta_F(G) + 1$ and $\chi_t(G) \geq \Delta(G) + 1$. Figure 2 depicts a generalized coloring of a graph $G$ using $\chi_t(G, F) = 3$ colors for the colored edge set $F = \{(v_1, v_2), (v_3, v_5), (v_3, v_7), (v_4, v_6), (v_5, v_6)\}$, where the uncolored edges $(v_1, v_4)$, $(v_2, v_7)$, $(v_3, v_6)$ and $(v_4, v_5)$ in $\bar{F}$ are drawn by dotted lines.

Suppose that $g$ is a generalized coloring of $G$ for $F$, $h$ is an ordinary edge-coloring of the subgraph $H = G[\bar{F}]$ of $G$ induced by $\bar{F}$, and $g$ and $h$ use disjoint sets of colors. Then, superimposing $h$ on $g$, one can obtain a total coloring $f$ of $G$. Unfortunately, the total coloring $f$ obtained in this way may use more than $\chi_t(G)$ colors even if $g$ uses $\chi_t(G, F)$ colors and $h$ uses

$\chi'(H)$ colors, because $\chi_t(G) \leq \chi_t(G,F) + \chi'(H)$ but the equality does not always hold; for example, $\chi_t(G) = 4$, $\chi_t(G,F) = 3$ and $\chi'(H) = 2$, and hence $\chi_t(G) < \chi_t(G,F) + \chi'(H)$ for the graph $G$ in Figure 2. However, in Section 3, we will show that, for a partial $k$-tree $G = (V,E)$ with the large maximum degree, there indeed exists $F \subseteq E$ such that $\chi_t(G) = \chi_t(G,F) + \chi'(H)$, and show that such a set $F$, a generalized coloring of $G$ for $F$ and an edge-coloring of $H$ can be found in linear time.

A graph $G = (V,E)$ is defined to be a *$k$-tree* if either it is a complete graph of $k$ vertices or it has a vertex $v \in V$ whose neighbors induce a clique of size $k$ and the graph $G - \{v\}$ obtained from $G$ by deleting the vertex $v$ and all edges incident to $v$ is again a $k$-tree. A graph is defined to be a *partial $k$-tree* if it is a subgraph of a $k$-tree [Bod90]. In the paper we assume that $k = O(1)$. The graph in Figure 1 is a partial 3-tree.

For a natural number $s$, an *$s$-numbering* of a graph $G = (V,E)$ is a bijection $\varphi : V \to \{1,2,\cdots,n\}$ such that

$$|\{(v,x) \in E : \varphi(v) < \varphi(x)\}| \leq s$$

for each vertex $v \in V$. A graph having an $s$-numbering is called an *$s$-degenerated graph.* Every partial $k$-tree $G$ is a $k$-degenerated graph, and its $k$-numbering can be found in linear time.

For an $s$-numbering $\varphi$ of $G$ and a vertex $v \in V$, we define

$$\begin{aligned}
E^{\mathrm{fw}}_{\varphi}(v,G) &= \{(v,x) \in E : \varphi(v) < \varphi(x)\};\\
E^{\mathrm{bw}}_{\varphi}(v,G) &= \{(x,v) \in E : \varphi(x) < \varphi(v)\};\\
d^{\mathrm{fw}}_{\varphi}(v,G) &= |E^{\mathrm{fw}}_{\varphi}(v,G)|; \text{ and}\\
d^{\mathrm{bw}}_{\varphi}(v,G) &= |E^{\mathrm{bw}}_{\varphi}(v,G)|.
\end{aligned}$$

The edges in $E^{\mathrm{fw}}_{\varphi}$ are called *forward edges*, and those in $E^{\mathrm{bw}}_{\varphi}$ *backward edges*. The definition of an $s$-numbering implies that $d^{\mathrm{fw}}_{\varphi}(v,G) \leq s$ for each vertex $v \in V$.

## 3 A Linear Algorithm

In this section we prove the following main theorem.

**Theorem 3.1** *Let $G = (V,E)$ be a partial $k$-tree with bounded $k$. Then there exists an algorithm to find a total coloring of $G$ with the minimum number $\chi_t(G)$ of colors in linear time.*

We first have the following lemma [ZNN96, IZN99].

**Lemma 3.2** *For any $s$-degenerated graph $G$, the following* (a) *and* (b) *hold:*

(a) *if $\Delta(G) \geq 2s$, then $\chi'(G) = \Delta(G)$;* and

(b) $\chi_t(G) \leq \Delta(G) + s + 2$.

Using a standard dynamic programming algorithm in [IZN99], one can solve the total coloring problem for a partial $k$-tree $G$ in time $O(n\chi_t^{2^{4(k+1)}})$ where $\chi_t = \chi_t(G)$; the size of a dynamic programming table updated by the algorithm is $O(\chi_t^{2^{4(k+1)}})$. Since $G$ is a partial $k$-tree, $G$ is $k$-degenerated. Furthermore $k = O(1)$. Therefore, if $\Delta(G) = O(1)$, then $\chi_t(G) = O(1)$ by Lemma 3.2(b) and hence the algorithm takes linear time to solve the total coloring problem. Thus it suffices to give an algorithm for the case $\Delta(G)$ is large, say $\Delta(G) \geq 8k^2$.

Our idea is to find a subset $F$ of $E$ such that $\chi_t(G) = \chi_t(G,F) + \chi'(H)$ as described in the following lemma.

**Lemma 3.3** *Assume that $G = (V,E)$ is an $s$-degenerated graph and has an $s$-numbering $\varphi$. If $\Delta(G) \geq 8s^2$, then there exists a subset $F$ of $E$ satisfying the following conditions* (a)–(h):

(a) *$\Delta(G) = \Delta_F(G) + \Delta_{\bar{F}}(G)$, where $\bar{F} = E - F$;*

(b) *$\Delta_F(G) \geq s + 1$;*

(c) *$\Delta_{\bar{F}}(G) \geq 2s$;*

(d) *the set $F$ can be found in linear time;*

(e) *$\varphi$ is a 1-numbering of $G' = (V,F)$, and hence $G'$ is a forest ;*

(f) *$\chi_t(G,F) = \Delta_F(G) + 1$, and a generalized coloring of $G$ for $F$ using $\Delta_F(G) + 1$ colors can be found in linear time;*

(g) *$\chi'(H) = \Delta_{\bar{F}}(G)$, where $H = (V,\bar{F})$;* and

(h) *$\chi_t(G) = \chi_t(G,F) + \chi'(H)$.*

**Proof.** The proofs of (a)–(e) will be given later. We now prove only (f)–(h).

(f) Let $C$ be a set of $\Delta_F(G) + 1$ colors. For each $i = 1,2,\cdots,n$, let $v_i$ be a vertex of $G$ such that $\varphi(v_i) = i$, let $N^{\mathrm{fw}}(v_i) = \{x \in V : (v_i,x) \in$

$E$, $\varphi(v_i) < \varphi(x)\}$, and let $E_F^{\rm fw}(v_i) = \{(v_i, x) \in F : \varphi(v_i) < \varphi(x)\}$. Since $\varphi$ is an $s$-numbering of $G$, $d_\varphi^{\rm fw}(v_i, G) = |N^{\rm fw}(v_i)| \le s$ for each $i = 1, 2, \cdots, n$. By (e) $\varphi$ is a 1-numbering of $G' = (V, F)$, and hence $d_\varphi^{\rm fw}(v_i, G') = |E_F^{\rm fw}(v_i)| \le 1$ for each $i = 1, 2, \cdots, n$.

We construct a generalized coloring $g$ of $G$ for $F$ using colors in $C$ as follows. We first color $v_n$ by any color $c$ in $C$: let $g(v_n) := c$. Suppose that we have colored the vertices $v_n, v_{n-1}, \cdots, v_{i+1}$ and the edges in $E_F^{\rm fw}(v_{n-1}) \cup E_F^{\rm fw}(v_{n-2}) \cup \cdots \cup E_F^{\rm fw}(v_{i+1})$, and that we are now going to color $v_i$ and the edge in $E_F^{\rm fw}(v_i)$ if $E_F^{\rm fw}(v_i) \ne \emptyset$. There are two cases to consider.

**Case 1**: $E_F^{\rm fw}(v_i) \ne \emptyset$.

In this case $E_F^{\rm fw}(v_i)$ contains exactly one edge $e = (v_i, v_j)$, where $i < j \le n$.

We first color $e$. Let $C' = \{g((v_j, v_l)) : (v_j, v_l) \in F,\ i+1 \le l \le n\} \subseteq C$, then we must assign to $e$ a color not in $\{g(v_j)\} \cup C'$. Since $e = (v_j, v_i) \in F$, we have

$$\begin{aligned} |\{(v_j, v_i)\} \cup \{(v_j, v_l) \in F : i+1 \le l \le n\}| \\ \le d(v_j, G') \end{aligned}$$

and hence $|C'| \le d(v_j, G') - 1$. Clearly $d(v_j, G') \le \Delta_F(G) = |C| - 1$. Therefore we have $|C'| \le |C| - 2$. Thus there exists a color $c' \in C$ not in $\{g(v_j)\} \cup C'$. We color $e$ by $c'$: let $g(e) := c'$.

We next color $v_i$. Let $C'' = \{g(x) : x \in N^{\rm fw}(v_i)\}$, then we must assign to $v_i$ a color not in $\{c'\} \cup C''$. Since $|C''| \le |N^{\rm fw}(v_i)| \le s$ and $\Delta_F(G) \ge s+1$ by (b) above, we have $|\{c'\} \cup C''| \le s+1 \le \Delta_F(G) = |C| - 1$. Thus there exists a color $c'' \in C$ not in $\{c'\} \cup C''$, and we can color $v_i$ by $c''$: let $g(v_i) := c''$.

**Case 2**: $E_F^{\rm fw}(v_i) = \emptyset$.

In this case we need to color only $v_i$. Similarly as above, there exists a color $c'' \in C$ not in $C''$ since $|C''| \le s < \Delta_F(G) < |C|$. Therefore we can color $v_i$ by $c''$: let $g(v_i) := c''$.

Thus we have colored $v_i$ and the edge in $E_F^{\rm fw}(v_i)$ if $E_F^{\rm fw}(v_i) \ne \emptyset$. Repeating the operation above for $i = n-1, n-2, \cdots, 1$, we can construct a generalized coloring $g$ of $G$ for $F$ using colors in $C$. Hence $\chi_t(G, F) \le |C| = \Delta_F(G) + 1$. Clearly $\chi_t(G, F) \ge \Delta_F(G) + 1$, and hence we have $\chi_t(G, F) = \Delta_F(G) + 1$. Clearly the construction of $g$ above takes linear time. Thus we have proved (f).

(g) Since $G$ is $s$-degenerated, the subgraph $H$ of $G$ is $s$-degenerated. By (c) we have $\Delta(H) = \Delta_{\bar F}(G) \ge 2s$. Therefore by Lemma 3.2(a) we have $\chi'(H) = \Delta(H) = \Delta_{\bar F}(G)$. Thus we have proved (g).

(h) We can obtain a total coloring of $G$ by superimposing an edge-coloring of $H$ on a generalized coloring of $G$ for $F$. Therefore we have $\chi_t(G) \le \chi_t(G, F) + \chi'(H)$. Since $\chi_t(G) \ge \Delta(G) + 1$, by (a), (f) and (g) we have

$$\begin{aligned} \chi_t(G) &\ge \Delta(G) + 1 \\ &= \Delta_F(G) + \Delta_{\bar F}(G) + 1 \\ &= \chi_t(G, F) + \chi'(H). \end{aligned}$$

Thus we have $\chi_t(G) = \chi_t(G, F) + \chi'(H)$.

*Q.E.D.*

We now have the following theorem.

**Theorem 3.4** *If $G$ is an $s$-degenerated graph and $\Delta(G) \ge 8s^2$, then $\chi_t(G) = \Delta(G) + 1$.*

**Proof.** By (a), (f), (g) and (h) in Lemma 3.3 we have

$$\begin{aligned} \chi_t(G) &= \chi_t(G, F) + \chi'(H) \\ &= \Delta_F(G) + 1 + \Delta_{\bar F}(G) \\ &= \Delta(G) + 1. \end{aligned}$$

We are now ready to present our algorithm to find a total coloring of a given partial $k$-tree $G = (V, E)$ with $\Delta(G) \ge 8k^2$.

**[Total-Coloring Algorithm]**

**Step 1.** Find a subset $F \subseteq E$ satisfying Conditions (a)–(h) in Lemma 3.3.

**Step 2.** Find a generalized coloring $g$ of $G$ for $F$ using $\chi_t(G, F) = \Delta_F(G) + 1$ colors.

**Step 3.** Find an ordinary edge-coloring $h$ of $H$ using $\chi'(H) = \Delta_{\bar F}(G)$ colors.

**Step 4.** Superimpose the edge-coloring $h$ on the generalized coloring $g$ to obtain a total coloring $f$ of $G$ using $\chi_t(G) = \Delta(G) + 1$ colors.

Since $G$ is a partial $k$-tree, $G$ is $k$-degenerated. Since $\Delta(G) \ge 8k^2$, by Lemma 3.3(d) one can find the subset $F \subseteq E$ in Step 1 in linear time. By Lemma 3.3(f) one can find the generalized coloring $g$ in Step 2 in linear time. Since $G$ is a partial $k$-tree, a subgraph $H$ of $G$ is also a partial $k$-tree. Therefore, in Step 3 one can find the edge-coloring $h$ of $H$ in linear time by an algorithm in [ZNN96] although $\chi'(H)$ is not always bounded. Thus the algorithm runs in linear time.

This completes the proof of Theorem 3.1.

In the remainder of this section we prove (a)–(e) of Lemma 3.3. We need the following two lemmas.

**Lemma 3.5** *Let $G=(V,E)$ be an $s$-degenerated graph, and let $\varphi$ be an $s$-numbering of $G$. If $\Delta(G)$ is even, then there exists a subset $E'$ of $E$ satisfying the following three conditions* (a)–(c):

(a) $\Delta(G') = \Delta(G'') = \Delta(G)/2$;

(b) *$\varphi$ is an $\lceil s/2 \rceil$-numbering of $G'$*; and

(c) $|E'| \le |E|/2$,

*where $G' = (V,E')$ and $G'' = (V, E-E')$. Furthermore, such a set $E'$ can be found in linear time.*

**Proof.** We first construct a new graph $G^*$ from $G$ as follows. For each vertex $v \in V$ of $G$, replace $v$ with two copies $v_{\rm fw}$ and $v_{\rm bw}$; attach to the copy $v_{\rm fw}$ the $d_\varphi^{\rm fw}(v,G)$ edges in $E_\varphi^{\rm fw}(v,G)$; if $d_\varphi^{\rm fw}(v,G)$ is even, then attach to the copy $v_{\rm bw}$ the $d_\varphi^{\rm bw}(v,G)$ edges in $E_\varphi^{\rm bw}(v,G)$; and if $d_\varphi^{\rm fw}(v,G)$ is odd, then attach to the copy $v_{\rm bw}$ any $d_\varphi^{\rm bw}(v,G)-1$ edges in $E_\varphi^{\rm bw}(v,G)$ and attach to the copy $v_{\rm fw}$ the remaining edge in $E_\varphi^{\rm bw}(v,G)$. Let $G^*$ be a resulting graph. Note that $d(v_{\rm fw},G^*)$ is even and $d(v,G) = d(v_{\rm fw},G^*) + d(v_{\rm bw},G^*)$ for each vertex $v \in V$.

We then construct a graph $G^{**}$ from $G^*$ as follows. Add a dummy vertex to $G^*$ and add dummy edges joining the vertex and every vertex of odd degree in $G^*$. Let $G^{**}$ be the resulting graph. Then each connected component of $G^{**}$ is Eulerian, and has an Eulerian circuit.

We number the edges in $G^{**}$ by $1,2,\cdots,|E(G^{**})|$ along any Eulerian circuits of the connected components of $G^{**}$. Let $E^*_{\rm od}$ be the set of odd-numbered (non-dummy) edges in $G^*$, and let $E^*_{\rm ev}$ be the set of even-numbered (non-dummy) edges in $G^*$. Let $G^*_{\rm od}$ be the subgraph of $G^*$ induced by $E^*_{\rm od}$, and let $G^*_{\rm ev}$ be the subgraph of $G^*$ induced by $E^*_{\rm ev}$. Then, since $d(v_{\rm fw},G^*)$ is even for any vertex $v \in V$, we have

$$d(v_{\rm fw},G^*_{\rm od}) = d(v_{\rm fw},G^*_{\rm ev}) = \frac{d(v_{\rm fw},G^*)}{2}. \qquad (1)$$

On the other hand, since $d(v_{\rm bw},G^*)$ is not always even and at most one dummy edge is incident to $v_{\rm bw}$, we have

$$\begin{aligned}\left\lfloor \frac{d(v_{\rm bw},G^*)}{2} \right\rfloor &\le d(v_{\rm bw},G^*_{\rm od}) \\ &\le \left\lceil \frac{d(v_{\rm bw},G^*)}{2} \right\rceil, \end{aligned} \qquad (2)$$

and

$$\begin{aligned}\left\lfloor \frac{d(v_{\rm bw},G^*)}{2} \right\rfloor &\le d(v_{\rm bw},G^*_{\rm ev}) \\ &\le \left\lceil \frac{d(v_{\rm bw},G^*)}{2} \right\rceil. \end{aligned} \qquad (3)$$

One may assume without loss of generality that $|E^*_{\rm ev}| \le |E^*_{\rm od}|$. Let $E'$ be the set of all edges of $G$ that correspond to edges in $E^*_{\rm ev}$ of $G^*$. We first show that (a) holds for this set $E'$. Let $v \in V$ be any vertex of $G$. Then by the construction of $G^*$ and $G^*_{\rm ev}$ above we have

$$d_{E'}(v,G) = d(v_{\rm fw},G^*_{\rm ev}) + d(v_{\rm bw},G^*_{\rm ev}). \qquad (4)$$

By Eqs. (1), (3) and (4) we have

$$d_{E'}(v,G) \le \frac{d(v_{\rm fw},G^*)}{2} + \left\lceil \frac{d(v_{\rm bw},G^*)}{2} \right\rceil.$$

Since $d(v_{\rm fw},G^*)$ is even, we have

$$\begin{aligned} d_{E'}(v,G) &\le \left\lceil \frac{d(v_{\rm fw},G^*) + d(v_{\rm bw},G^*)}{2} \right\rceil \\ &= \left\lceil \frac{d(v,G)}{2} \right\rceil \\ &\le \left\lceil \frac{\Delta(G)}{2} \right\rceil, \end{aligned}$$

and hence $d_{E'}(v,G) \le \Delta(G)/2$ since $\Delta(G)$ is even. In particular, if $d(v,G) = \Delta(G)$, then we have $d_{E'}(v,G) = \Delta(G)/2$, because by Eqs. (1) and (3)

$$\begin{aligned} d_{E'}(v,G) &= d(v_{\rm fw},G^*_{\rm ev}) + d(v_{\rm bw},G^*_{\rm ev}) \\ &\ge \frac{d(v_{\rm fw},G^*)}{2} + \left\lfloor \frac{d(v_{\rm bw},G^*)}{2} \right\rfloor \\ &= \left\lfloor \frac{d(v_{\rm fw},G^*) + d(v_{\rm bw},G^*)}{2} \right\rfloor \\ &= \left\lfloor \frac{d(v,G)}{2} \right\rfloor \\ &= \frac{\Delta(G)}{2}. \end{aligned}$$

Thus we have $\Delta(G') = \Delta_{E'}(G) = \Delta(G)/2$. Similarly, we have $\Delta(G'') = \Delta(G)/2$. Thus we have shown that (a) holds.

We next show that (b) holds. We first claim that $d(v_{\rm fw}, G^*_{\rm ev}) \le \lceil s/2 \rceil$ for each vertex $v \in V$. Since $\varphi$ is an $s$-numbering of $G$, $d^{\rm fw}_{\varphi}(v,G) \le s$. Therefore $d(v_{\rm fw}, G^*) \le d^{\rm fw}_{\varphi}(v,G) + 1 \le s+1$, because all edges in $E^{\rm fw}_{\varphi}(v,G)$ and at most one edge in $E^{\rm bw}_{\varphi}(v,G)$ are attached to $v_{\rm fw}$ in the construction of $G^*$ above. Thus by Eq. (1) we have

$$d(v_{\rm fw}, G^*_{\rm ev}) = \frac{d(v_{\rm fw}, G^*)}{2} \le \frac{s+1}{2},$$

and hence $d(v_{\rm fw}, G^*_{\rm ev}) \le \lceil s/2 \rceil$ since both $s$ and $d(v_{\rm fw}, G^*_{\rm ev})$ are integers. Thus we have proved that $d(v_{\rm fw}, G^*_{\rm ev}) \le \lceil s/2 \rceil$. The edges in $G' = (V,E)$ correspond to the edges in $G^*_{\rm ev}$, and all edges in $E^{\rm fw}_{\varphi}(v,G')$ are attached to $v_{\rm fw}$ in $G^*_{\rm ev}$. Therefore $d^{\rm fw}_{\varphi}(v,G') \le d(v_{\rm fw}, G^*_{\rm ev})$. Hence $d^{\rm fw}_{\varphi}(v,G') \le \lceil s/2 \rceil$, and consequently $\varphi$ is an $\lceil s/2 \rceil$-numbering of $G'$.

Finally we have $|E'| = |E^*_{\rm ev}| \le |E|/2$ since $|E| = |E^*_{\rm ev}| + |E^*_{\rm od}|$ and $|E^*_{\rm ev}| \le |E^*_{\rm od}|$. $\mathcal{Q.E.D.}$

**Lemma 3.6** *Let $G = (V,E)$ be an $s$-degenerated graph, and let $\alpha$ be a natural number. If $\Delta(G) \ge 2\alpha \ge 2s$, then there exists a subset $E'$ of $E$ such that $\Delta_{E'}(G) + \Delta_{\bar{E}'}(G) = \Delta(G)$ and $\Delta_{E'}(G) = \alpha$ where $\bar{E}' = E - E'$. Furthermore, such a set $E'$ can be found in linear time.*

**Proof.** Since $G$ is an $s$-degenerated graph and $\Delta(G) \ge 2\alpha \ge 2s$, there exists a partition $\{E_1, E_2, \cdots, E_l\}$ of $E$ satisfying the following three conditions (i)–(iii) [ZNN96, pp. 610]:

(i) $\sum_{i=1}^{l} \Delta_{E_i}(G) = \Delta(G)$;

(ii) $\Delta_{E_i}(G) = \alpha$ for each $i = 1, 2, \cdots, l-1$; and

(iii) $\alpha \le \Delta_{E_l}(G) < 2\alpha$.

Let $E' = E_1$. Then $\Delta_{E'}(G) = \Delta_{E_1}(G) = \alpha$ and by (ii), clearly

$$\Delta_{\bar{E}'}(G) \ge \Delta(G) - \Delta_{E'}(G). \qquad (5)$$

On the other hand, since $\bar{E}' = E_2 \cup E_3 \cup \cdots \cup E_l$, by (i) we have

$$\begin{aligned} \Delta_{\bar{E}'}(G) &\le \sum_{i=2}^{l} \Delta_{E_i}(G) \\ &= \Delta(G) - \Delta_{E_1}(G) \\ &= \Delta(G) - \Delta_{E'}(G). \qquad (6) \end{aligned}$$

Thus by Eqs. (5) and (6) we have $\Delta_{\bar{E}'}(G) = \Delta(G) - \Delta_{E'}(G)$, and hence $\Delta_{E'}(G) + \Delta_{\bar{E}'}(G) = \Delta(G)$. Since the partition $\{E_1, E_2, \cdots, E_l\}$ of $E$ can be found in linear time [ZNN96], the set $E' = E_1$ can be found in linear time. $\mathcal{Q.E.D.}$

We are now ready to give the remaining proof of Lemma 3.3.

**Remaining Proof of Lemma 3.3**: Since we have already proved (f)–(h) before, we now prove (a)–(e).

Let $p = \lfloor \log \Delta(G) \rfloor$. Then $2^p \le \Delta(G) < 2^{p+1}$. Since $\Delta(G) \ge 8s^2$, we have $p = \lfloor \log \Delta(G) \rfloor \ge 3 + \lfloor 2\log s \rfloor > 2 + 2\log s$. Therefore we have $\Delta(G) \ge 2^p > 2^{2+2\log s} = 4s^2 > 2s$.

Let $q = \lceil \log s \rceil$. Then $2^{q-1} < s \le 2^q$. We find $F$ by constructing a sequence of $q+1$ spanning subgraphs $G_0, G_1, \cdots, G_q$ of $G$ as follows.

**procedure FIND-F**

**begin**

by Lemma 3.6, find a subset $E_0$ of $E$ such that

(3-1) $\Delta_{E_0}(G) = 2^{p-1}$; and

(3-2) $\Delta_{E_0}(G) + \Delta_{\bar{E}_0}(G) = \Delta(G)$, where $\bar{E}_0 = E - E_0$;

{Choose $\alpha = 2^{p-1}$,then $\Delta(G) \ge 2^p = 2\alpha \ge 2s$, and hence there exists such a set $E_0$ by Lemma 3.6}

let $G_0 := (V, E_0)$ and let $s_0 := s$; {$\Delta(G_0) = 2^{p-1}$ and $\varphi$ is an $s_0$-numbering of $G_0$}

**for** $i := 0$ to $q-1$ **do**

**begin**

by Lemma 3.5, find a subset $E'_i$ of $E_i$ satisfying

(7-1) $\Delta(G'_i) = \Delta(G''_i) = \Delta(G_i)/2$; {$\Delta(G_i)$ is even}

(7-2) $\varphi$ is an $s_{i+1}$-numbering of $G'_i$, where $s_{i+1} = \lceil s_i/2 \rceil$; and

(7-3) $|E'_i| \le |E_i|/2$,

where $G'_i = (V, E'_i)$, $G''_i = (V, E''_i)$ and $E''_i = E_i - E'_i$;

```
8      let E_{i+1} := E'_i and let G_{i+1} := (V, E_{i+1});
9      end
10  let F := E_q;
11  end.
```

We first prove (a). Since $F = E_q$, $\Delta_F(G) = \Delta_{E_q}(G) = \Delta(G_q)$. Therefore we have

$$\Delta_{\bar{F}}(G) \geq \Delta(G) - \Delta_F(G) = \Delta(G) - \Delta(G_q). \quad (7)$$

By line 7 and line 8 in the procedure above, we have $G_{i+1} = G'_i$, $\Delta(G_{i+1}) = \Delta(G'_i)$ and $\Delta(G'_i) + \Delta(G''_i) = \Delta(G_i)$, and hence $\Delta(G''_i) = \Delta(G_i) - \Delta(G_{i+1})$ for each $i = 0, 1, \cdots, q-1$. Therefore we have

$$\begin{aligned} \sum_{i=0}^{q-1} \Delta(G''_i) &= \sum_{i=0}^{q-1} (\Delta(G_i) - \Delta(G_{i+1})) \\ &= \Delta(G_0) - \Delta(G_q). \end{aligned} \quad (8)$$

Since $\Delta(G_0) = \Delta_{E_0}(G)$, by (3-2) in the procedure above we have

$$\Delta(G_0) + \Delta_{\bar{E}_0}(G) = \Delta(G). \quad (9)$$

Furthermore, since $\bar{F} = E - F = \bar{E}_0 \cup E''_0 \cup E''_1 \cup \cdots \cup E''_{q-1}$ and $\Delta_{E''_i}(G) = \Delta(G''_i)$ for each $i = 0, 1, \cdots, q-1$, by Eqs. (8) and (9) we have

$$\begin{aligned} \Delta_{\bar{F}}(G) &\leq \Delta_{\bar{E}_0}(G) + \sum_{i=0}^{q-1} \Delta_{E''_i}(G) \\ &= \Delta_{\bar{E}_0}(G) + \sum_{i=0}^{q-1} \Delta(G''_i) \\ &= \Delta_{\bar{E}_0}(G) + \Delta(G_0) - \Delta(G_q) \\ &= \Delta(G) - \Delta(G_q). \end{aligned} \quad (10)$$

Therefore by Eqs. (7) and (10) we have $\Delta_{\bar{F}}(G) = \Delta(G) - \Delta(G_q)$. Since $\Delta(G_q) = \Delta_F(G)$, we have $\Delta_{\bar{F}}(G) = \Delta(G) - \Delta_F(G)$, and hence $\Delta_F(G) + \Delta_{\bar{F}}(G) = \Delta(G)$. Thus we have proved (a).

We next prove (b). By (7-1) and line 8 in the procedure above we have $\Delta(G_{i+1}) = \Delta(G'_i) = \Delta(G_i)/2$ for each $i = 0, 1, \cdots, q-1$, and by (3-1) we have $\Delta(G_0) = 2^{p-1}$. Therefore we have

$$\Delta_F(G) = \Delta(G_q) = \frac{\Delta(G_0)}{2^q} = 2^{p-q-1}. \quad (11)$$

Since $2^p > 4s^2$ and $2^{q-1} < s$, we have $\Delta_F(G) = 2^{p-q-1} > 4s^2/4s = s$. Thus we have $\Delta_F(G) \geq s+1$, and hence (b) holds.

We next prove (c). By (a) and Eq. (11) we have $\Delta_{\bar{F}}(G) = \Delta(G) - \Delta_F(G) = \Delta(G) - 2^{p-q-1}$, and hence

$$\begin{aligned} \Delta_{\bar{F}}(G) &= \Delta(G) - \frac{2^p}{2^{q+1}} \\ &\geq \Delta(G) - \frac{\Delta(G)}{2^{q+1}} \\ &= \Delta(G)(1 - \frac{1}{2^{q+1}}) \\ &\geq 8s^2(1 - \frac{1}{2s}) \\ &= 4s(2s-1) \\ &\geq 2s \end{aligned}$$

since $\Delta(G) \geq 8s^2$, $\Delta(G) \geq 2^p$ and $s \leq 2^q$. Thus we have proved (c).

We next prove (d). By Lemma 3.6, line 3 can be done in time $O(|E|)$. By Lemma 3.5, line 7 can be done in time $O(|E_i|)$. Therefore the **for** statement in line 5 can be done in time $O(\sum_{i=0}^{q-1} |E_i|)$ time. Since $|E_0| \leq |E|$ and $|E_{i+1}| = |E'_i| \leq |E_i|/2$ for each $i = 0, 1, \cdots, q-1$ by (7-3) in the procedure above, we have $\sum_{i=0}^{q-1} |E_i| \leq 2|E|$. Thus one can know that the **for** statement can be done in time $O(|E|)$. Thus $F$ can be found in linear time, and hence (d) holds.

We finally prove (e). Since $s = s_0 \leq 2^q$ and $s_i = \lceil s_{i-1}/2 \rceil \leq s_{i-1}/2 + 1/2$ for each $i = 1, 2, \cdots, q$, we have

$$\begin{aligned} s_q &\leq \frac{s_0}{2^q} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^q} \\ &= \frac{s_0}{2^q} + 1 - \frac{1}{2^q} \\ &< 2, \end{aligned}$$

and hence $s_q = 1$. Therefore $\varphi$ is a 1-numbering of $G' = (V, F)$. Thus we have proved (e).

This completes the proof of Lemma 3.3.

## 4 Conclusion

In this paper we showed that the edge-disjoint paths problem is NP-complete for partial 3-trees. Since the graph $G_f$ constructed by our reduction has a bounded pathwidth, our reduction implies that the edge-disjoint paths problem is NP-complete for the class of graphs with bounded pathwidth. Therefore the maximum edge-disjoint paths problem is NP-hard for the same class. It should be noted that the graph constructed by the reduction in [GVY97] has an unbounded pathwidth.

Zhou *et al.* proved the following fact: the edge-disjoint paths problem can be solved in polynomial time for a partial $k$-tree $G$ if the augmented graph $G^+$ obtained from $G$ by adding $p$ edges $(s_i, t_i)$, $1 \leq i \leq p$, remains to be a partial $k$-tree [ZTN96]. The result in this paper does not conflict with the fact above, because the augmented graph $G_f^+$ of $G_f$ is not always a partial $k$-tree with bounded $k$.

A class of tractable problems for partial $k$-trees has been characterized in terms of the monadic second order logic [ACPS93, ALS91, BPT92, Cou90]. It remains open to characterize a class of intractable problems, including the edge-disjoint paths problem, the subgraph isomorphism problem, and the bandwidth problem.

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